KILLARA HIGH SCHOOL
MATHEMATICS DEPARTMENT

Student Name

**2022** **YEAR 12**

# Mathematics Extension 2

Trial HSC Examination

Date: Monday 8th August, 2022

| Q | Marks |
|---|---|
| MC | /10 |
| 11 | /17 |
| 12 | /14 |
| 13 | /15 |
| 14 | /14 |
| 15 | /17 |
| 16 | /13 |
| **Total** | **/100** |
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**General Instructions:**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

**Total Marks:** 100

**Section I - 10 marks**

- Allow about 15 minutes for this section

**Section II - 90 marks**

- Allow about 2 hours and 45 minutes for this section

**This question paper must not be removed from the examination room.**

*This assessment task constitutes 30% of the course.*

## Section I

**10 marks**

**Allow about 15 minutes for this section**

Use the multiple-choice sheet for Questions 1–10

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**1** Consider the statement:

"If $2^n - 1$ is a prime, then $n$ is a prime."

The contrapositive of the statement would be written as:

(A) If $2^n - 1$ is not a prime, then $n$ is not a prime.

(B) If $n$ is not a prime, then $2^n - 1$ is a prime.

(C) If $n$ is not a prime, then $2^n - 1$ is not a prime.

(D) If $2^n - 1$ is a prime, then $n$ is not a prime.

**2** The negation of the statement "$\exists$ a real number $x$ such that $x^2 = 9$ or $x > 2$" is best given by:

(A) $\forall$ real numbers $x, x^2 \neq 9$ or $x \leq 2$

(B) $\exists$ a real number $x, x^2 \neq 9$ and $x \leq 2$

(C) $\exists$ a real number $x, x^2 \neq 9$ or $x \leq 2$

(D) $\forall$ real numbers $x, x^2 \neq 9$ and $x \leq 2$

**3** If $z = 2e^{\frac{i\pi}{6}}$, then $\mathrm{Re}(z^3 - z)$ is:

(A) $-\sqrt{3}$

(B) $\sqrt{3}$

(C) $8 - \sqrt{3}$

(D) $7\frac{1}{2}$

**4** A particle moves in a straight line with a distance of $x$ metres from the origin and a speed of $v$ metres per second. It is given that $v = \sqrt{12 - x^2}$ and the initial displacement is $x = 3$. Which of the following is an expression for $t$?

(A) $24x + 2x^3 - 126$

(B) $6\sin^{-1}\left(\frac{x}{2\sqrt{3}}\right) + \frac{x}{2}\sqrt{12 - x^2} - 2\pi - \frac{3\sqrt{3}}{2}$

(C) $\dfrac{\sin^{-1} x - \sin^{-1} 3}{2\sqrt{3}}$

(D) $\sin^{-1}\frac{x}{2\sqrt{3}} - \frac{\pi}{3}$

**Examination continues on next page**

**5** Given that $z - \sqrt[4]{-i} = 0$, then $z$ could be:

(A) $i$

(B) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

(C) $\cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8}$

(D) $e^{\frac{i\pi}{8}}$

**6** By graphical means or otherwise, which integral has the smallest value?

(A) $\int_0^{\frac{\pi}{6}} \sin^2 x\, dx$

(B) $\int_0^{\frac{\pi}{6}} (1 - \sin x) dx$

(C) $\int_0^{\frac{\pi}{6}} (1 - \sin^2 x) dx$

(D) $\int_0^{\frac{\pi}{6}} (1 - \sin x)^2 dx$

**7** Let $\underset{\sim}{a} = 14\underset{\sim}{i} + 4\underset{\sim}{j} + \alpha^2\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} - 2\underset{\sim}{j} + \alpha\underset{\sim}{k}$, where $\alpha$ is a real constant.

If the length of the projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{14}{\sqrt{3}}$, then $\alpha$ equals:

(A) 1

(B) 2

(C) 3

(D) 4

**8** A particle is projected from level ground at an angle to the horizontal and experiences a resistive force opposed to the direction of motion.

Which of the following statements is **FALSE**?

(A) The magnitude of the angle of impact is greater than the magnitude of the angle of projection.

(B) The magnitude of the horizontal component of velocity is larger during the ascent of the particle than the descent.

(C) The range of the particle would be greater if it had the same velocity and angle of projection but a smaller mass.

(D) More than half of the horizontal range is travelled during the first half of the flight time.

**Examination continues on next page**

**9** The integral

$$\int_1^3 \frac{\cos^3\left(\frac{\pi}{8}x\right)}{x(4-x)}dx$$

is equivalent to:

(A) $\displaystyle\int_1^3 \frac{\sin^3\left(\frac{\pi}{8}u\right)}{u(4-u)}du$

(B) $\displaystyle\int_1^3 \frac{\sin^3\left(\frac{\pi}{8}u\right)}{u(u-4)}du$

(C) $\displaystyle\int_1^3 \frac{\cos^3\left(-\frac{\pi}{8}u\right)}{u(u-4)}du$

(D) $\displaystyle\int_1^3 \frac{\sin^3\left(-\frac{\pi}{8}u\right)}{u(4-u)}du$

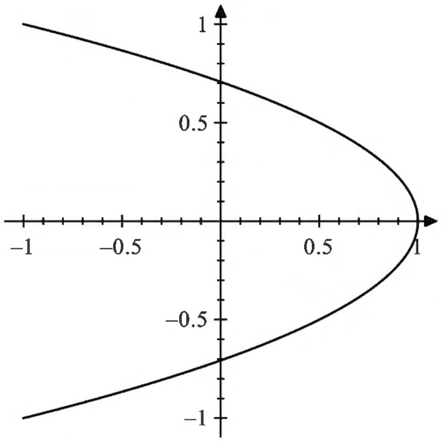
**10** The diagrams below show graphs in a two-dimensional plane.

When viewed in the $xy$, $xz$ or $yz$ planes, which of the following diagrams is a possible view of the curve defined by the vector $\underset{\sim}{r}$ below?

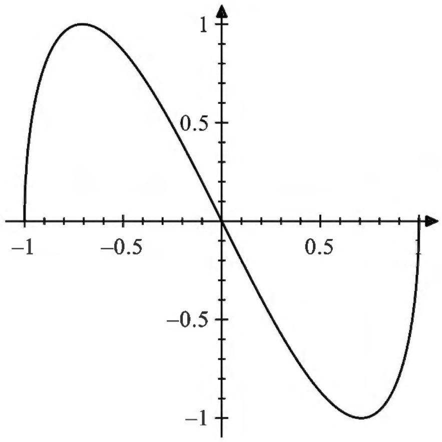
$$\underset{\sim}{r} = (\cos 2t)\underset{\sim}{i} + (\cos t)\underset{\sim}{j} + (\sin 2t)\underset{\sim}{k}$$

$$\text{for } 0 \le t \le \pi$$

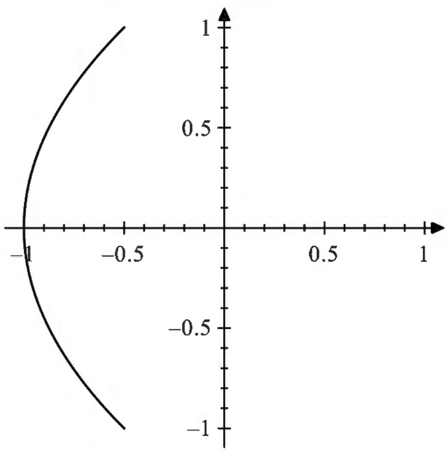
(A)



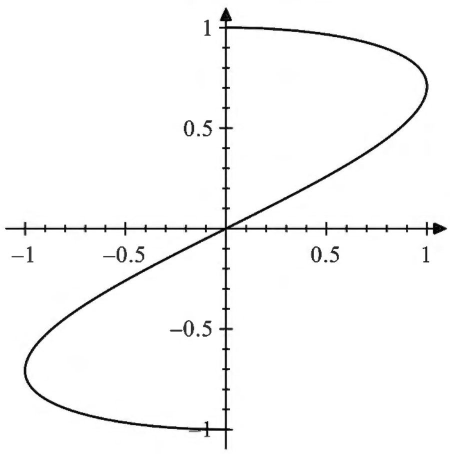

(B)




(C)




(D)




**End of Section I**

**Examination continues on next page**

## Section II

**90 marks**

**Allow about 2 hours and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11-16, your response should include relevant mathematical reasoning and/or calculations.

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**Question 11 (**17 Marks) Use the Question 11 Writing Booklet.

(a) Consider the complex numbers $z = \sqrt{2}\left(1 + i\sqrt{3}\right)$ and $w = \sqrt{6}(1 + i)$

(i) Find $z - w$. **1**

(ii) Express $\frac{z}{w}$ in the form $a + ib$. **1**

(iii) Express $z$ and $w$ in modulus-argument form. **1**

(iv) Hence, find the exact value of $\cos\frac{\pi}{12}$. **2**

(b) (i) Find: **2**

$$\int x\cos(x^2)\,dx.$$

(ii) Find: **3**

$$\int \frac{1}{4+5\cos x}\,dx$$

(iii) Find: **3**

$$\int \frac{5}{(x-1)(x^2+1)}\,dx$$

(c) Consider the three points $A\ (4,1,-1)$, $B\ (5,2,0)$ and $C\ (-1,-1,2)$.

(i) Show that the points are not collinear. **1**

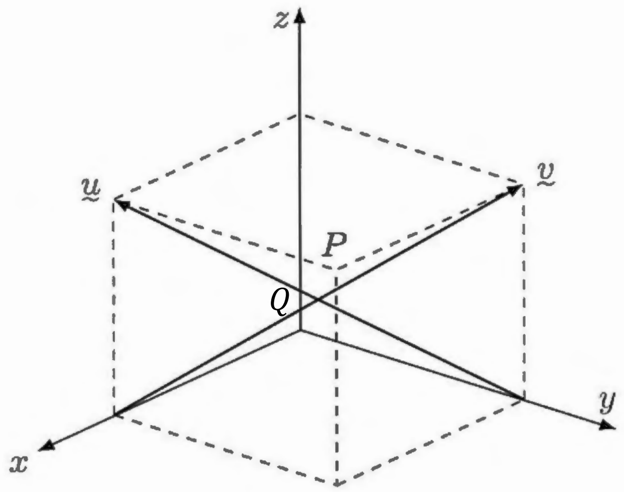
(ii) Find the size of $\angle ABC$ to the nearest minute. **2**

(iii) Hence, or otherwise, find the area of the triangle $\Delta ABC$ to 3 decimal places. **1**

**Examination continues on next page**

**Question 12 (**14 Marks) Use the Question 12 Writing Booklet.

(a) The rectangular prism shown below is produced by the origin and the point $P(a, b, c)$ with all sides either within the coordinate planes or parallel to them.




The main diagonals are represented by the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ as shown, intersecting at point $Q$.

(i) Find expressions for $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in terms of $a$, $b$ and $c$. **1**

(ii) Show that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular if and only if $a^2 + b^2 = c^2$. **2**

(iii) Find a point $Q$ such that the main diagonals are perpendicular. **1**

(b) Evaluate the following: **3**

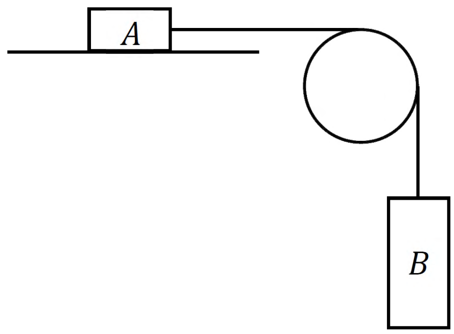
$$\int_{-\frac{5}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{x^2 + 2\sqrt{3}x + 7} dx$$

(c) The velocity of a particle is given by $\dot{x} = x^3 - 2x$ metres per second. What is the acceleration of the particle when $x = 1$? **2**

(d) Prove that for $a, b \in \mathbb{Z}$, if $a$ divides $b$ with no remainder then $a^2$ divides $b^2$ with no remainder. **2**

(e) Two blocks are connected by an inextensible string passing over a frictionless pulley. Block $A$ has mass $m$ kg and lies on a smooth horizontal surface. It is connected to Block $B$ with mass $4m$ kg that hangs vertically at the other end of the string. Both blocks are initially at rest.

The system is set in motion and let $x$ be the distance travelled to the right by Block $A$ after $t$ seconds.




(i) Construct free body diagrams to show the forces acting upon Blocks $A$ and $B$. **1**

(ii) Find an expression for $x$ in terms of $g$ and $t$. **2**

**Examination continues on next page**

**Question 13 (**15 Marks) Use the Question 13 Writing Booklet.

(a) (i) On an argand diagram, sketch the region defined by $\frac{\pi}{4} \leq \arg\left(\frac{z-1}{z+i}\right) \leq \pi$. **3**

(ii) Find the minimum and maximum values of $|z|$. **2**

(b) Prove $|a^2 - ab| + |ab - b^2| \geq (a+b)(a-b)$ for $a > b > 0$. **2**

(c) Consider the two spheres with equations given below.

$$\left|\underset{\sim}{r} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}\right| = 5$$

$$\left|\underset{\sim}{r} - \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix}\right| = 10$$

(i) Show that the spheres have a single point of contact. **1**

(ii) Find the coordinates of the point of contact. **2**

(d) Prove that the sum of two integers is even if and only if they have the same parity. That is, both integers are either both odd or both even. **3**

(e) Use integration by parts to find: **2**

$$\int e^x \sin 2x \, dx$$

**Question 14** (14 Marks) Use the Question 14 Writing Booklet.

(a) A rocket with initial velocity $u\ ms^{-1}$ is projected vertically upwards from the surface of a planet. The acceleration due to gravity is $\frac{k}{x^4}\ ms^{-2}$ towards the centre of the planet, where $x$ is the distance in metres from the centre of the planet and $k \in \mathbb{R}^+$. Assume air resistance is negligible.

(i) Show that $k = gr^4$, where $r$ is the radius of the planet in metres and $g\ ms^{-2}$ is the acceleration due to gravity at the surface of the planet. **1**

(ii) Given that the speed of the rocket at a distance $x$ metres from the centre is $v\ ms^{-1}$, show that: **2**

$$v^2 = u^2 + \frac{2k}{3}\left(\frac{1}{x^3} - \frac{1}{r^3}\right)$$

(iii) Give an expression for the escape velocity in terms of $g$ and $r$. **2**

(iv) If $u^2 = \frac{2gr}{3}$, find the time taken for the rocket to travel to a point $3r$ metres above the planet's surface. Give your answer in terms of $g$ and $r$. **2**

(v) If $u^2 = \frac{gr}{3}$, find the distance travelled before the rocket returns to the planet. Give your answer in terms of $r$. **2**

**Examination continues on next page**

(b) Let: **3**

$$I_n = \int_1^e x(\ln x)^n \, dx$$

where $n = 0, 1, 2, 3, \ldots$

Using integration by parts, show that:

$$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}, \qquad n = 1, 2, 3, \ldots$$

(c) The area bounded by the curve $y = \sqrt{x}(\ln x), x \geq 1$, the $x$-axis and the line $x = e$ is rotated about the $x$-axis through $2\pi$ radians. Find the exact value of the volume of the solid formed. **2**

**Question 15** (17 Marks) Use the Question 15 Writing Booklet.

(a) Let the points $E$, $F$, $G$ and $H$ have corresponding position vectors from some origin $O$ given below:

$$\overrightarrow{OE} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}, \overrightarrow{OF} = \begin{pmatrix} 6 \\ -4 \\ -5 \end{pmatrix}, \overrightarrow{OG} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{OH} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

(i) Show that the line: **2**

$$\underset{\sim}{m} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

describes points that are equidistant from $E$ and $F$, for some real constant $\alpha$.

(ii) The line GH has the equation: **3**

$$\underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

for some real constant $\beta$.

(You do not need to prove this).

Show that the lines $\underset{\sim}{m}$ and $\underset{\sim}{n}$ are skew.

(iii) Hence, find shortest distance between the lines. **3**

**Examination continues on next page**

(b) A submarine of mass $m$ is propelled in a straight line underwater by a force $F$. The water exerts a resistive force of magnitude $kv$ against the direction of motion of the submarine, where $v$ is the submarine's speed and $k$ is a constant. Initially the submarine starts from rest.

(i) Show that the submarine reaches half its limiting speed when $t = \frac{m}{k}\ln 2$. **3**

(ii) When the submarine reaches the speed $\frac{F}{2k}$, the captain decides to reverse the engines so that the propelling force $F$ is now acting backwards to slow the ship down. Show that it would cover a further distance of $\frac{mF}{2k^2}\left(1 - 2\ln\frac{3}{2}\right)$ before coming to rest. **3**

(c) An integer $N > 91$ is divisible by 13 if when the unit digit is multiplied by 9 and subtracted from the rest of the number, the outcome is also divisible by 13. For example, 858 is divisible by 13 as $85 - 8 \times 9 = 13$, which is divisible by 13. **3**

Prove this divisibility theorem for the general case of a four digit number with digits $abcd$, noting that the place value of the leading digit '$a$' is $1000 \times a$.

**Question 16** (13 Marks) Use the Question 16 Writing Booklet.

(a) (i) Show that for $z = \cos\theta + i\sin\theta$: **1**

$$z^n + (\bar{z})^n = 2\cos(n\theta)$$

and

$$z^n - (\bar{z})^n = 2i\sin(n\theta)$$

(ii) Show that: **2**

$$(\cot\theta + i)^n - (\cot\theta - i)^n = \frac{2i\sin(n\theta)}{\sin^n\theta}$$

(iii) Show that the roots of the equation $(x+i)^n - (x-i)^n = 0$ are: **3**

$$\cot\frac{\pi}{n}, \cot\frac{2\pi}{n}, \cot\frac{3\pi}{n}, \dots, \cot\frac{(n-1)\pi}{n}$$

(iv) Show that: **1**

$$(x+i)^n - (x-i)^n = 2i\left(\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3} + \binom{n}{5}x^{n-5} - \binom{n}{7}x^{n-7} \dots\right)$$

(v) Hence, show that: **2**

$$\cot^2\frac{\pi}{n} + \cot^2\frac{2\pi}{n} + \cot^2\frac{3\pi}{n} + \dots + \cot^2\frac{(n-1)\pi}{n} = \frac{(n-1)(n-2)}{3}$$

**Examination continues on next page**

(b) A projectile is launched with velocity $v$ ms$^{-1}$ at an angle of $\alpha$ to the horizontal up an inclined plane of inclination $\beta$, where $\alpha > \beta$.

The equations of motion of the projectile are given as:

$$\underset{\sim}{r} = \begin{pmatrix} vt\cos\alpha \\ vt\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$$

(Do not prove this)

(i) Show that the path of the projectile is given as: **1**

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$$

(ii) Prove that the projectile will hit the surface of the plane at a right angle when: **3**

$$\tan\alpha = \cot\beta + 2\tan\beta$$

**End of Examination**

# 2022 KHS Ext 2 Task 4 Trial – Solutions, Marking Criteria and Marker's Feedback

| Q | Worked Solutions | Marking Criteria | Marker's Feedback |
|---|---|---|---|
| 1 | C | | |
| 2 | D | | |
| 3 | A | | |
| 4 | D | | |
| 5 | C | | |
| 6 | A | | |
| 7 | B | | |
| 8 | C | | |
| 9 | A | | |
| 10 | D | | |
| 11ai | $z - w = \sqrt{2} - \sqrt{6} + (\sqrt{6} - \sqrt{6})i$<br>$= \sqrt{2} - \sqrt{6}$ | 1 mark answer or equivalent | |
| ii | $\frac{z}{w} = \frac{\sqrt{2} + \sqrt{6}i}{\sqrt{6} + \sqrt{6}i} \times \frac{\sqrt{6} - \sqrt{6}i}{\sqrt{6} - \sqrt{6}i}$<br>$= \frac{\sqrt{12} + 6 + (6 - \sqrt{12})i}{6 + 6}$<br>$= \frac{6 + 2\sqrt{3}}{12} + \frac{6 - 2\sqrt{3}}{12}i$<br>$= \frac{3 + \sqrt{3}}{6} + \frac{3 - \sqrt{3}}{6}i$ | 1 mark answer or equivalent | Too many marks lost here for careless calculation. Check your work so you don't lose easy marks. |
| iii | $z = 2\sqrt{2}cis\left(\frac{\pi}{3}\right)$<br>$w = 2\sqrt{3}cis\left(\frac{\pi}{4}\right)$ | 1 mark both answers or equivalent | |
| iv | $\frac{z}{w} = \frac{2\sqrt{2}cis\left(\frac{\pi}{3}\right)}{2\sqrt{3}cis\left(\frac{\pi}{4}\right)}$<br>$= \frac{\sqrt{2}}{\sqrt{3}}cis\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$<br>$= \frac{\sqrt{6}}{3}cis\left(\frac{\pi}{12}\right)$<br>$= \frac{3+\sqrt{3}}{6} + \frac{3-\sqrt{3}}{6}i$ from part (ii) | 1 mark for dividing<br><br>1 mark for equating real components and finding a result | |

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| | Equating real components: $$\frac{\sqrt{6}}{3}\cos\frac{\pi}{12}=\frac{3+\sqrt{3}}{6}$$ $$\cos\frac{\pi}{12}=\frac{3+\sqrt{3}}{2\sqrt{6}}$$ $$=\frac{1+\sqrt{3}}{2\sqrt{2}}$$ $$=\frac{\sqrt{2}+\sqrt{6}}{4}$$ | | |
| bi | $$I=\int x\cos(x^2)\,dx$$ $$=\frac{1}{2}\int 2x\cos(x^2)\,dx$$ $$=\frac{1}{2}\sin(x^2)+c$$ | 1 mark for integrating trigonometric function<br><br>1 mark for correct coefficient and constant of integration | Reverse chain rule. |
| ii | Let $t=\tan\frac{x}{2}$ $$dx=\frac{2}{1+t^2}dt$$ $$I=\int\frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)}\times\frac{2}{1+t^2}dt$$ $$=2\int\frac{1}{4+4t^2+5-5t^2}dt$$ $$=2\int\frac{1}{9-t^2}dt$$ $$=2\int\frac{1}{(3-t)(3+t)}dt$$ $=2\int\frac{1}{6}\left(\frac{1}{3-t}+\frac{1}{3+t}\right)dt$ (partial fractions) $$=\frac{1}{3}(-\ln\lvert 3-t\rvert+\ln\lvert 3+t\rvert)+c$$ $$=\frac{1}{3}\ln\left\lvert\frac{3+t}{3-t}\right\rvert+c$$ $$=\frac{1}{3}\ln\left(\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right)+c$$ | 1 mark for correctly applying t substitution<br><br>1 mark for partial fractions<br><br>1 mark for correct integral | Some students factorised "2" before partial fractions BUT also included it in the calculation of partial fractions.<br><br>Quite a few did not remember to substitute back to $x$. REMEMBER: If you are using sub for integral, sub back!<br><br>This is a good example where the absolute value of natural log integration is required. |

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| iii | $$I = 5\int \frac{1}{(x-1)(x^2+1)}dx$$ Let$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+}{x^2+1}$ $$1 = Ax^2 + A + Bx^2 + Cx - Bx - C$$ $$1 = (A+B)x^2 + (C-B)x + A - C$$ $$A + B = 0$$ $$C - B = 0$$ $$A - C = 1$$ $$A = \frac{1}{2}$$ $$B = -\frac{1}{2}$$ $$C = -\frac{1}{2}$$ $$I = \frac{5}{2}\int \frac{1}{x-1} - \frac{x+1}{x^2+1}dx$$ $$= \frac{5}{2}\int \frac{1}{x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}dx$$ $$= \frac{5}{2}\left(\ln\lvert x-1\rvert - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x\right) + c$$ | 1 mark for correct partial fractions<br><br>1 mark for integral to two logarithms<br><br>1 mark for integral to inverse tan | |
| ci | $$\overrightarrow{BA} = \begin{pmatrix}4\\1\\-1\end{pmatrix} - \begin{pmatrix}5\\2\\0\end{pmatrix}$$ $$= \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$ $$\overrightarrow{BC} = \begin{pmatrix}-1\\-1\\2\end{pmatrix} - \begin{pmatrix}5\\2\\0\end{pmatrix}$$ $$= \begin{pmatrix}-6\\-3\\2\end{pmatrix}$$ Assume the points are collinear, i.e. $\overrightarrow{BA} = \lambda\,\overrightarrow{BC}$<br>Considering $z$ components: $$-1 = \lambda \times 2$$ $$\lambda = -\frac{1}{2}$$ Considering $y$ components | 1 mark for showing contradiction or equivalent method | You need to SHOW the contradiction rather than just state it. Remember, one of the topics is specifically on PROOF so don't just assume it is obvious. The mark was paid generously in this instance, however, if this was a 2 mark question not many students would have received full marks. |

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| | $-1 = \lambda \times -3$<br>$\lambda = \frac{1}{3}$<br>$\neq -\frac{1}{2}$<br>This is a contradiction and hence the assumption must be false, and the points are not collinear. | | |
| ii | $\overrightarrow{BA} \cdot \overrightarrow{BC} = 6 + 3 - 2$<br>$= 7$<br>$\left\lvert\overrightarrow{BA}\right\rvert = \sqrt{1+1+1}$<br>$= \sqrt{3}$<br>$\left\lvert\overrightarrow{BC}\right\rvert = \sqrt{36+9+4}$<br>$= 7$<br>$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left\lvert\overrightarrow{BA}\right\rvert \left\lvert\overrightarrow{BC}\right\rvert}$<br>$= \frac{7}{7\sqrt{3}}$<br>$\theta = \cos^{-1}\frac{1}{\sqrt{3}}$<br>$= 54°44'$ | 1 mark for dot product<br><br>1 mark for finding angle | Tail to tail for angle between. |
| iii | $Area = \frac{1}{2}\left\lvert\overrightarrow{BA}\right\rvert \left\lvert\overrightarrow{BC}\right\rvert \sin 54°44'$<br>$= 4.950\ u^2$ | 1 mark for area<br>ECF allowed | Surprising number of responses had the formula wrong. It is on the formula sheet. Advanced maths material is fundamental here! |
| 12ai | $\underset{\sim}{u} = \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} - \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$<br>$= \begin{pmatrix} a \\ -b \\ c \end{pmatrix}$<br>$\underset{\sim}{v} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} - \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$<br>$= \begin{pmatrix} -a \\ b \\ c \end{pmatrix}$ | 1 mark for both correct vectors | |

| | | | |
|---|---|---|---|
| ii | $$\underset{\sim}{u}\cdot\underset{\sim}{v}=0$$ $$\begin{pmatrix}a\\-b\\c\end{pmatrix}\cdot\begin{pmatrix}-a\\b\\c\end{pmatrix}=0$$ $$-a^2-b^2+c^2=0$$ $$a^2+b^2=c^2$$ | 1 mark for applying dot product<br>1 mark for simplifying | |
| iii | $a, b$ and $c$ must satisfy a Pythagorean triad, e.g. $$a=3$$ $$b=4$$ $$c=5$$ $$\overrightarrow{OQ}=\frac{1}{2}\overrightarrow{OP}$$ $$=\frac{1}{2}\begin{pmatrix}3\\4\\5\end{pmatrix}$$ $$=\begin{pmatrix}\frac{3}{2}\\2\\\frac{5}{2}\end{pmatrix}$$ Hence, point $Q$ could be $\left(\frac{3}{2},2,\frac{5}{2}\right)$. | 1 mark for any correct point where coordinates are half of Pythagorean triad | |
| b | $$I=\int_{-\frac{5}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{\left(x+\sqrt{3}\right)^2+2^2}dx$$ $$=\frac{1}{2}\left[\tan^{-1}\frac{x+\sqrt{3}}{2}\right]_{-\frac{5}{\sqrt{3}}}^{\sqrt{3}}$$ $$=\frac{1}{2}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$$ $$=\frac{1}{2}\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$$ $$=\frac{\pi}{4}$$ | 1 mark for for factorising denominator<br><br>1 mark for integrating to inverse tan<br><br>1 mark for correctly applying boundaries and simplifying | |
| c | $$\dot{x}=v=x^3-2x$$ $$\frac{dv}{dx}=3x^2-2$$ $$\ddot{x}=a=v\frac{dv}{dx}=(x^3-2x)(3x^2-2)$$ When $x=1$ $$a=(1-2)(3-2)$$ | 1 mark for taking derivative of $v$<br><br>1 mark for finding result | |

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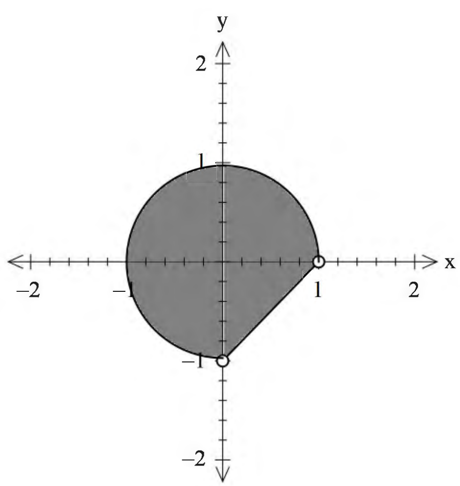
| | $= -1\ ms^{-1}$ | | |
| d | $a$ divides $b$ implies $b = na$ for some integer $n$.<br>Then $b^2 = b \times b = (na)(na)$ by our proposition<br>$= n^2(a^2)$<br><br>Rearranging we get:<br>$$\frac{b^2}{a^2} = n^2$$<br>That is, $a^2$ divides $b^2$ | 1 mark for establishing proposition and algebra towards $b^2$<br>2 marks for expressing result and conclusion | |
| ei | Diagrams | 1 mark for two correct diagrams including tension, force due to gravity and normal force | Many forgot tension forces |
| ii | $$(m+4m)a = 4mg$$ $$a = \frac{4}{5}g$$ $$v = \frac{4}{5}gt$$ $$x = \frac{2}{5}gt^2$$ | 1 mark for correctly defining system of motion<br><br>1 mark for integrating correctly w.r.t. time | |
| 13ai |   | 1 mark interval, including excluded points<br><br>1 mark for outer bound (not intercepts)<br><br>1 mark for region | Poorly done<br>Intercepts often missed and some not shaded |
| ii | $|z|_{min} = 0$ | 1 mark for min | |

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| | Angle at circumference is $\frac{\pi}{4}$, hence angle at centre is a right angle. This places the centre at the origin, and gives the circle a radius of 1.<br>$$\lvert z\rvert_{max} = 1$$ | 1 mark for max, providing reasoning for centre | |
| b | $\lvert a^2 - ab\rvert + \lvert ab - b^2\rvert \geq \lvert a^2 - ab + ab - b^2\rvert$ by the triangle inequality<br>$\lvert a^2 - ab\rvert + \lvert ab - b^2\rvert \geq \lvert a^2 - b^2\rvert$<br>$\lvert a^2 - ab\rvert + \lvert ab - b^2\rvert \geq \lvert (a+b)(a-b)\rvert$<br>And as $a > b > 0$ we know:<br>$a + b > 0$ and $a - b > 0$, so<br>$\lvert a^2 - ab\rvert + \lvert ab - b^2\rvert \geq (a+b)(a-b)$ | 1 mark correct use of the triangle inequality<br>2 marks correctly resolving answer given $a > b > 0$ | Many used the result |
| ci | For a single point of contact, the distance between centres must equal the sum of the radii<br>$$Distance\ between\ centres = \left\lvert \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \right\rvert$$<br>$$= \left\lvert \begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix} \right\rvert$$<br>$$= \sqrt{100 + 121 + 4}$$<br>$$= \sqrt{225}$$<br>$$= 15$$<br>$$= 10 + 5$$<br>$$= sum\ of\ the\ radii$$ | 1 mark for proof with equivalent reasoning | Done well |
| ii | Let the centres be $C_1(3, -4, 3)$ and $C_2(-7, 7, 1)$ such that:<br>$$\overrightarrow{C_1C_2} = \begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix}$$<br>Point of contact occurs at $C_1$ shifted by the vector $\frac{5}{15}\overrightarrow{C_1C_2}$<br>$$= \frac{1}{3}\begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix}$$<br>$$= \begin{pmatrix} -\frac{10}{3} \\ \frac{11}{3} \\ -\frac{2}{3} \end{pmatrix}$$<br>Hence, the point of contact occurs at $\left(-\frac{1}{3}, -\frac{1}{3}, \frac{7}{3}\right)$ | 1 mark for consider the vector between centres<br><br>1 mark for finding coordinates | Poorly done<br>Often used projection poorly |

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| d | If two numbers are both even then they can be expressed as $a = 2m$ and $b = 2n$ for $m, n$ integer.<br><br>Hence $a + b = 2m + 2n = 2(m + n)$ which is even<br><br>If the two numbers are odd, the can be expressed as $a = 2m + 1$ and $b = 2n + 1$ for $m, n$ integer.<br><br>Hence $a + b = 2m + 1 + 2n + 1 = 2(m + n + 1)$ which is even.<br><br>However, if the integers do not have same parity they can be expressed as $a = 2m$ and $b = 2n + 1$<br><br>Hence $a + b = 2m + 2n + 1 = 2(m + n) + 1$ which is odd.<br><br>And so the sum of two integers is even iff they have the same parity. | 1 mark for a logical and reasoned plan, like cases<br>2 marks for a logical and reasoned plan correctly executed by missing a logical step, like the odd plus even case<br>3 marks a complete and correct proof with a conclusion. | If mistakes were made, it was not checking the different parity case for iff. |
| e | $I = \int e^x \sin 2x \, dx$<br>$= e^x \sin 2x - 2\int e^x \cos 2x \, dx$<br>$= e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x \, dx\right)$<br>$= e^x(\sin 2x - 2\cos 2x) - 4I$<br>$5I = e^x(\sin 2x - 2\cos 2x) + C$<br>$I = \dfrac{e^x(\sin 2x - 2\cos 2x)}{5} + C$ | 1 mark for correct first application of integration by parts<br><br>1 mark for result | Done well |
| 14ai | $ma = \dfrac{mk}{x^4}$<br>$k = ax^4$<br>When $x = r, a = g$<br>$k = gr^4$ | 1 mark for defining system of motion and substituting | $g = \frac{k}{x^4}$ is untrue. This implies $g$ is variable instead of a constant. |
| ii | $a = v\dfrac{dv}{dx} = -\dfrac{gr^4}{x^4}$<br>$\int_u^v v \, dv = -k\int_r^x x^{-4} \, dx$<br>$\frac{1}{2}[v^2]_u^v = \frac{k}{3}[x^{-3}]_r^x$<br>$\frac{1}{2}(v^2 - u^2) = \frac{k}{3}\left(\frac{1}{x^3} - \frac{1}{r^3}\right)$<br>$v^2 = u^2 + \frac{2k}{3}\left(\frac{1}{x^3} - \frac{1}{r^3}\right)$ | 1 mark for separating to integrate w.r.t. x<br><br>1 mark for integrating using bounds or finding constant and simplifying | Careful with negatives when defining the system. |

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| iii | $$v^2 = u^2 + \frac{2k}{3}\left(\frac{1}{x^3} - \frac{1}{r^3}\right)$$ As $x \to \infty, \frac{1}{x^3} \to 0$ $$v^2 = u^2 - \frac{2k}{3r^3}$$ For $v^2 > 0$ $$u^2 > \frac{2k}{3r^3}$$ $$u > \sqrt{\frac{2k}{3r^3}}, u > 0$$ Therefore, as $k = gr^4$, escape velocity is when: $$u > \sqrt{\frac{2gr}{3}}\ ms^{-1}$$ | 1 mark for applying $\frac{1}{x^3} \to 0$<br><br>1 mark for finding result with working | Some getting $x \to \infty$<br>Barely any taking $v^2 > 0$ |
| iv | $$v^2 = \frac{2gr}{3} + \frac{2k}{3}\left(\frac{1}{x^3} - \frac{1}{r^3}\right)$$ $$= \frac{2gr}{3} + \frac{2gr^4}{3x^3} - \frac{2gr}{3}$$ $$= \frac{2gr^4}{3x^3}$$ $$v = \frac{dx}{dt} = \frac{r^2}{x^{\frac{3}{2}}}\sqrt{\frac{2g}{3}}$$ $$t = \frac{1}{r^2}\sqrt{\frac{3}{2g}}\int_r^{4r} x^{\frac{3}{2}}\,dx$$ $$= \frac{1}{r^2}\sqrt{\frac{3}{2g}} \times \frac{2}{5}\left[x^{\frac{5}{2}}\right]_r^{4r}$$ $$= \frac{1}{r^2}\sqrt{\frac{3}{2g}} \times \frac{2}{5}\left(32r^{\frac{5}{2}} - r^{\frac{5}{2}}\right)$$ $$= \frac{31}{5}\sqrt{\frac{6r}{g}}\ seconds$$ | 1 mark for separating and integrating w.r.t. time<br><br>1 mark for applying correct bounds or finding constant to get result | Not many achieved these marks.<br>Careful algebra required.<br>Note displacement is to $4r$ not $3r$ |

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| v | $$v^2 = \frac{gr}{3} + \frac{2k}{3}\left(\frac{1}{x^3} - \frac{1}{r^3}\right)$$ $$= \frac{2gr^4}{3x^3} - \frac{gr}{3}$$ Max height occurs when $v = 0$ $$\frac{2gr^4}{3x^3} - \frac{gr}{3} = 0$$ $$x^3 = 2r^3$$ $$x = \sqrt[3]{2}r$$ Distance travelled $= 2 \times (max\ height\ - r)$ $$x = 2r(\sqrt[3]{2} - 1)\ metres$$ | 1 mark for applying $v = 0$<br><br>1 mark for finding distance | |
| b | $$I_n = \int_1^e x(\ln x)^n\, dx$$ $$= \frac{1}{2}[x^2(\ln x)^n]_1^e - \int_1^e \frac{x^2}{2} \times \frac{n(\ln x)^{n-1}}{x} dx$$ $$= \frac{1}{2}(e^2 \ln e - 1^2 \ln 1) - \frac{n}{2}\int_1^e x\,(\ln x)^{n-1} dx$$ $$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$ | 1 mark for integrating by parts<br><br>1 mark for correct bounds and simplifying<br><br>1 mark for simplifying to recurrence relation with working | |
| c | $$V = \pi \int_1^e y^2\, dx$$ $$= \pi \int_1^e x\,(\ln x)^2 dx$$ $= \pi \frac{e^2}{2} - \frac{2}{2}\pi I_1$ from (b) $$= \frac{e^2\pi}{2} - \pi \int_1^e x \ln x\, dx$$ $$= \frac{e^2\pi}{2} - \pi\left(\frac{e^2}{2} - \frac{1}{2} I_0\right)$$ $$= \frac{\pi}{2}\int_1^e x\, dx$$ $$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_1^e$$ $$= \frac{e^2 - 1}{4}\pi\ units^2$$ | 1 mark for finding volume as a correct integral<br><br>1 mark for result | Please check formulas from ext 1 material. |

<table>
<tr><td>15ai</td><td>$$E=(-2,4,3)$$ $$F=(6,-4,-5)$$ Let M be midpoint between $E$ and $F$ $$M=\left(\frac{-2+6}{2},\frac{4-4}{2},\frac{3-5}{2}\right)$$ $$=(2,0,-1)$$ $$\overrightarrow{OM}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$$ For points on the line to be equidistant from $E$ and $F$, direction vector must be perpendicular to $\overrightarrow{EF}$<br><br>$$\overrightarrow{EF}=\begin{pmatrix}8\\-8\\-8\end{pmatrix}$$ $$\begin{pmatrix}8\\-8\\-8\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=8+0-8$$ $$=0$$ Hence, $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is perpendicular to $\overrightarrow{EF}$ and points on the line $\underset{\sim}{m}=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\1\end{pmatrix}$ are equidistant from $\overrightarrow{EF}$</td><td>1 mark for midpoint<br><br>1 mark for direction vector, with sufficient reasoning</td><td></td></tr>
<tr><td>ii</td><td>Skew lines are not parallel and do not intersect<br>For parallel lines, direction vectors must be parallel.<br>i.e. $\begin{pmatrix}1\\0\\1\end{pmatrix}=\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$<br>Considering $x$ components: $$1=\lambda\times-1$$ $$\lambda=-1$$ Considering $y$ components: $$0=2\lambda$$ $$\lambda=0$$ $$\neq-1$$ Hence, the direction vectors are not paralle and the lines are not parallel.<br><br>For point of intersection: $$\begin{pmatrix}2\\0\\-1\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}+\beta\begin{pmatrix}-1\\2\\1\end{pmatrix}$$</td><td>1 mark for dealing with parallel lines, with reasoning<br><br>1 mark for equating lines<br><br>1 mark for finding contradiction</td><td>Mostly well done</td></tr>
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| | This gives the following equations: $$2+\alpha = 1-\beta \ (1)$$ $$0 = -2+2\beta \ (2)$$ $$-1+\alpha = -2+\beta \ (3)$$ From (2) $$\beta = 1$$ Sub in (1) $$2+\alpha = 1-1$$ $$\alpha = -2$$ Check in (3) $$LHS = -1-2$$ $$= -3$$ $$RHS = -2+1$$ $$= -1$$ $$\neq LHS$$ Hence, the lines do not intersect.<br>Thus, the lines are skew. | | |
| iii | Consider the points $P$ on $\underset{\sim}{m}$ and $Q$ on $\underset{\sim}{n}$: $$P = (2+\alpha, 0, -1+\alpha)$$ $$Q = (1-\beta, -2+2\beta, -2+\beta)$$ $$\overrightarrow{PQ} = \begin{pmatrix} -1-\beta-\alpha \\ -2+2\beta \\ -1+\beta-\alpha \end{pmatrix}$$ For shortest distance, $\overrightarrow{PQ}$ must be perpendicular to both $\underset{\sim}{m}$ and $\underset{\sim}{n}$ $$\begin{pmatrix} -1-\beta-\alpha \\ -2+2\beta \\ -1+\beta-\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$ $$-1-\beta-\alpha-1+\beta-\alpha = 0$$ $$\alpha = -1$$ $$\overrightarrow{PQ} = \begin{pmatrix} -1-\beta+1 \\ -2+2\beta \\ -1+\beta+1 \end{pmatrix}$$ $$= \begin{pmatrix} -\beta \\ -2+2\beta \\ \beta \end{pmatrix}$$ | 1 mark for finding $\overrightarrow{PQ}$ or equivalent<br><br>1 mark for finding $\alpha$ and $\beta$ or equivalent<br><br>1 mark for distance | Less well done.<br>Little reasoning provided |

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| | $\begin{pmatrix}-\beta\\-2+2\beta\\\beta\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=0$<br>$\beta-4+4\beta+\beta=0$<br>$\beta=\frac{2}{3}$<br>$\overrightarrow{PQ}=\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$<br>Minimum distance<br>$\left\|\overrightarrow{PQ}\right\|=\left\|\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{pmatrix}\right\|$<br>$=\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}}$<br>$=\frac{2\sqrt{3}}{3}$ | | |
| bi | $ma=m\frac{dv}{dt}=F-kv$<br>$t=-\frac{m}{k}\int_0^v-\frac{k}{F-kv}dv$<br>$=-\frac{m}{k}[\ln\|F-kv\|]_0^v$<br>$=-\frac{m}{k}\ln\left\|\frac{F-kv}{F}\right\|$<br>$=-\frac{m}{k}\ln\left\|1-\frac{kv}{F}\right\|$<br>$t\to\infty$ as$1-\frac{kv}{F}\to 0$ i.e. $\frac{kv}{F}\to 1$<br>Therefore, limiting speed is $v_T=\frac{F}{k}$<br>$\frac{1}{2}v_T=\frac{F}{2k}$ | 1 mark for system of motion and integrating w.r.t. $v$<br><br>1 mark for finding limit as $t\to\infty$<br><br>1 mark for result with sufficient working | |

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| | $$t = -\frac{m}{k}\ln\left\|1 - \frac{k\frac{F}{2k}}{F}\right\|$$ $$= -\frac{m}{k}\ln\frac{1}{2}$$ $$= \frac{m}{k}\ln 2$$ | | |
| ii | $$ma = mv\frac{dv}{dx} = -F - kv$$ $$\int_0^x dx = -m\int_{\frac{F}{2k}}^0 \frac{v}{F+kv}dv$$ $$x = -\frac{m}{k}\int_{\frac{F}{2k}}^0 \frac{F+kv-F}{F+kv}dv$$ $$= -\frac{m}{k}\int_{\frac{F}{2k}}^0 1 - \frac{F}{F+kv}dv$$ $$= -\frac{m}{k}\int_{\frac{F}{2k}}^0 1 - \frac{F}{k}\times\frac{k}{F+kv}dv$$ $$= -\frac{m}{k}\left[v - \frac{F}{k}\ln\|F+kv\|\right]_{\frac{F}{2k}}^0$$ $$= -\frac{m}{k}\left(\left(-\frac{F}{k}\ln F\right) - \left(\frac{F}{2k} - \frac{F}{k}\ln\left(F + \frac{F}{2}\right)\right)\right)$$ $$= \frac{mF}{k^2}\ln F + \frac{mF}{2k^2} - \frac{mF}{k^2}\left(\ln\frac{3F}{2}\right)$$ $$= \frac{mF}{2k^2}\left(1 + 2\ln F - 2\ln\left(\frac{3F}{2}\right)\right)$$ $$= \frac{mF}{2k^2}\left(1 + 2\ln F - (2\ln F + 2\ln\frac{3}{2}\right)$$ $$= \frac{mF}{2k^2}\left(1 - 2\ln\frac{3}{2}\right)$$ | 1 mark for system of motion and defining integral<br><br>1 mark for integrating w.r.t. $v$<br><br>1 mark for applying bounds and finding result with sufficient working | Done well.<br>Some setting out concerns |
| c | For the number $N = abcd = 1000a + 100b + 10c + d$<br>We know that $(100a + 10b + c) - 9d = 13k$ for some integer $k$. $-(1)$<br><br>Now:<br>$$N = 1000a + 100b + 10c + d$$ $$N = 10(100a + 10b + c) + d$$ | 1 mark writing initial given in algebra<br>2 marks resolving N in terms of initial given result | Very limited argument, lots of algebra. Not great, needs more reasoning. |

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| | From (1), $(100a + 10b + c) = 13k + 9d$<br>$N = 10(13k + 9d) + d$<br>$= 130k + 91d$<br>$= 13(10k + 7d)$<br>which is divisible by 13. | 3 marks using initial given result to successfully express N as 13m and hence conclude it is divisible by 13, with reasoning. | |
| 16ai | $z^n = \cos(n\theta) + i\sin(n\theta)$<br>$(\bar{z})^n = \overline{(z^n)} = \cos(n\theta) - i\sin(n\theta)$<br><br>$z^n + (\bar{z})^n = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$<br>$= 2\cos(n\theta)$<br><br>$z^n - (\bar{z})^n = \cos(n\theta) + i\sin(n\theta) - (\cos(n\theta) - i\sin(n\theta))$<br>$= 2i\sin(n\theta)$ | 1 mark for result with sufficient working | |
| Ii | $(\cot\theta + i)^n = \left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\sin\theta}i\right)^n$<br>$= \frac{1}{\sin^n\theta}(\cos(n\theta) + i\sin(n\theta))$<br>Similarly,<br>$(\cot\theta - i)^n = \left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\sin\theta}i\right)^n$<br>$= \frac{1}{\sin^n\theta}(\cos(n\theta) - i\sin(n\theta))$<br><br>$(\cot\theta + i)^n - (\cot\theta - i)^n = \frac{1}{\sin^n\theta}\big(\cos(n\theta) + i\sin(n\theta) - (\cos(n\theta) - i\sin(n\theta))\big)$<br>$= \frac{1}{\sin^n\theta}2i\sin(n\theta)$ | 1 mark for factorising $\sin^n\theta$<br><br>1 mark for relating to part (i) and finding result with sufficient working | |
| Iii | When $x = \cot\theta$,<br>Solutions occur when $2i\sin(n\theta) = 0 + 0i$,<br>$\sin(n\theta) = 0$<br>$n\theta = 0, \pi, 2\pi, \dots, (n-1)\pi$<br>$\theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots, \frac{(n-1)\pi}{n}$<br>But $\sin^n\theta \neq 0$ i.e. $\theta \neq 0$<br>$\theta = \frac{\pi}{n}, \frac{2\pi}{n}, \dots, \frac{(n-1)\pi}{n}$ | 1 mark for equation in $\sin(n\theta)$<br><br>1 mark for solutions to $\theta$<br><br>1 mark for solutions for $x$, including reasoning for excluding $\cot 0$ | |

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| | Hence, solutions are $x = \cot\frac{\pi}{n}, \cot\frac{2\pi}{n}, \ldots, \cot\frac{(n-1)\pi}{n}$<br><br>ALTERNATE METHOD<br><br>$(x+i)^n = (x-i)^n$<br>$\left(\frac{x+i}{x-i}\right)^n = 1$<br>$\frac{x+i}{x-i} = e^{i\alpha}, where\ \alpha = \frac{2k\pi}{n}, k = 0, 1, 2, 3, \ldots, n-1$<br>$x + i = xe^{i\alpha} - ie^{i\alpha}$<br>$x - xe^{i\alpha} = -ie^{i\alpha} - i$<br>$x(1 - e^{i\alpha}) = -i(e^{i\alpha} + 1)$<br>$x = i\frac{e^{i\alpha}+1}{e^{i\alpha}-1} \div \frac{e^{\frac{i\alpha}{2}}}{e^{\frac{i\alpha}{2}}}$<br>$= i\frac{e^{\frac{i\alpha}{2}} + e^{-\frac{i\alpha}{2}}}{e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}}}$<br>$= i\frac{2\cos\frac{\alpha}{2}}{2i\sin\frac{\alpha}{2}}$<br>$= \cot\frac{\alpha}{2}$<br>$= \cot\left(\frac{k\pi}{n}\right)$ | 1 mark for finding equation to power of $n$<br><br>1 mark for finding solution to $n$th root with $x$ as the subject<br><br>1 mark for simplifying and finding result | |
| iv | Using binomial expansion<br>$(x+i)^n = x^n + \binom{n}{1}x^{n-1}i - \binom{n}{2}x^{n-2} - \binom{n}{3}x^{n-3}i + \binom{n}{4}x^{n-4} \ldots (1)$<br>$(x-i)^n = x^n - \binom{n}{1}x^{n-1}i - \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3}i + \binom{n}{4}x^{n-4} \ldots (2)$<br><br>$(1) - (2)$<br>$(x+i)^n - (x-i)^n = x^n + \binom{n}{1}x^{n-1}i - \binom{n}{2}x^{n-2} - \binom{n}{3}x^{n-3}i + \binom{n}{4}x^{n-4} \ldots$<br>$-x^n + \binom{n}{1}x^{n-1}i + \binom{n}{2}x^{n-2} - \binom{n}{3}x^{n-3}i - \binom{n}{4}x^{n-4} \ldots$<br>$= 2\binom{n}{1}x^{n-1}i - 2\binom{n}{3}x^{n-3}i + 2\binom{n}{5}x^{n-5}i - 2\binom{n}{7}x^{n-7}i \ldots$<br>$= 2i\left(\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3} + \binom{n}{5}x^{n-5} - \binom{n}{7}x^{n-7} \ldots\right)$ | 1 mark for showing binomial expansions with sufficient working | This type of result needs a lot of care when writing the expansion clearly.<br><br>You need to give sufficient terms to demonstrate pattern. |

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| v | Note that:<br>$\textit{Sum of the square of the roots}$<br>$= \textit{square of the sum of the roots} - 2 \times \textit{sum of the roots two at a time}$<br>$$\cot^2\frac{\pi}{n} + \cot^2\frac{2\pi}{n} + \cot^2\frac{3\pi}{n} + \cdots + \cot^2\frac{(n-1)\pi}{n} = 0 - 2 \times \frac{-\binom{n}{3}}{\binom{n}{1}}$$ $$= 2\frac{\frac{n!}{3!\,(n-3)!}}{\frac{n!}{1!\,(n-1)!}}$$ $$= 2\frac{(n-1)!}{3!\,(n-3)!}$$ $$= \frac{(n-1)(n-2)}{3}$$ | 1 mark for noting expression for sum of the square of the roots<br><br>1 mark for evaluating and simplifying | Poorly done. Need to review roots of polynomials. |
| bi | $$x = vt\cos\alpha$$ $$t = \frac{x}{v\cos\alpha}\ (1)$$ $$y = vt\sin\alpha - \frac{1}{2}gt^2\ (2)$$ $$\textit{Sub } (1) \textit{ in } (2)$$ $$y = \frac{vx\sin\alpha}{v\cos\alpha} - \frac{1}{2}g\left(\frac{x^2}{v^2\cos^2\alpha}\right)$$ $$= x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$$ | 1 mark for eliminating parameter and finding result with sufficient working | |
| ii | Path of projectile:<br>$$y_{proj} = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$$ $$y'_{proj} = \tan\alpha - \frac{gx\sec^2\alpha}{v^2}$$ Equation of inclined plane<br>$$y_{pla} = x\tan\beta$$ $$y'_{pla} = \tan\beta$$ Point of intersection occurs when $y_{proj} = y_{pla}$<br>$$x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2} = x\tan\beta$$ $$x\left(\frac{gx\sec^2\alpha}{2v^2} + (\tan\beta - \tan\alpha)\right) = 0$$ | 1 mark for substituting to find point of intersection<br><br>1 mark for taking derivatives as negative reciprocals<br><br>1 mark for substituting and finding result | Very poorly done, although potentially this was due to running out of energy at the end of the test. Students should have achieved a stronger result for this question given the relative difficulty. |

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| | For $x \neq 0$<br>$$\frac{gx\sec^2\alpha}{2v^2} + (\tan\beta - \tan\alpha) = 0$$<br>$$x = \frac{2v^2(\tan\alpha - \tan\beta)\cos^2\alpha}{g} \ (3)$$<br>For intersection at right angle $y'_{proj} = -\frac{1}{y'_{pla}}$<br>$$\tan\alpha - \frac{gx\sec^2\alpha}{v^2} = -\cot\beta$$<br>Sub (3)<br>$$\tan\alpha - \frac{g\sec^2\alpha}{v^2} \times \frac{2v^2(\tan\alpha - \tan\beta)\cos^2\alpha}{g} = -\cot\beta$$<br>$$\tan\alpha - 2(\tan\alpha - \tan\beta) = -\cot\beta$$<br>$$2\tan\beta - \tan\alpha = -\cot\beta$$<br>$$\tan\alpha = \cot\beta + 2\tan\beta$$ | | |